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A Sufficient Condition for the Rapid Convergence of Differential Approximation

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INTRODUCTION

Although there are a number of algorithms for finding parameters a_{nj}, λ_{nj}, $j = 1, \ldots, n$, which make the exponential sum

$$Y_n(t) = a_{n1} E_{\lambda_{n1}}(t) + \cdots + a_{nn} E_{\lambda_{nn}}(t), \tag{1}$$

where

$$E_\lambda(t) = e^{-\lambda t} \tag{2}$$

a "good" if not best least-squares approximation to a given real valued function F on $[0, \infty)$, cf. [1, 4–6], there is no corresponding analysis of the rate of convergence of Y_n to F as $n \to \infty$. In this paper we show that for almost all of these algorithms

$$\|F - Y_n\|_2 \leqslant K \cdot q^n, \qquad n = 1, 2, \ldots, \tag{3}$$

where K is a positive constant, $0 < q < 1$, and $\| \ \|_2$ is the usual L_2 norm on $[0, \infty)$ provided that F is a completely monotonic function having the representation

$$F(t) = \int_{\lambda=\alpha}^{\beta} e^{-\lambda t}\, df(\lambda), \qquad 0 < \alpha < \beta < \infty, \quad t \geqslant 0, \tag{4}$$

where df is a finite nonnegative measure.

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Some Auxiliary Functions

Given distinct positive numbers $\lambda_1,..., \lambda_n$ and $\lambda > 0$ we define

$$\mathscr{E}_\lambda(\lambda_1,..., \lambda_n; t) = a_1(\lambda_1,..., \lambda_n, \lambda)\, E_{\lambda_1}(t) + \cdots + a_n(\lambda_1,..., \lambda_n, \lambda)\, E_{\lambda_n}(t), \qquad t \geqslant 0, \tag{5}$$

to be the best least-squares approximation to (2) on $[0, \infty)$ from the n dimensional linear space spanned by $E_{\lambda_1},..., E_{\lambda_n}$.

Lemma. *The error in the approximation of E_λ by $\mathscr{E}_\lambda(\lambda_1,..., \lambda_n; -)$ has the norm*

$$\| E_\lambda - \mathscr{E}_\lambda(\lambda_1,..., \lambda_n; -)\|_2 = (2\lambda)^{-1/2} \cdot |D(\lambda)|, \tag{6}$$

where

$$D(\lambda) = \prod_{j=1}^{n} |(\lambda - \lambda_j)/(\lambda + \lambda_j)|. \tag{7}$$

Proof. Using Gram's lemma [2, p. 194] we find

$$\| E_\lambda - \mathscr{E}_\lambda(\lambda_1,..., \lambda_n; -)\|_2^2 = G(\lambda_1,..., \lambda_n, \lambda)/G(\lambda_1,..., \lambda_n), \tag{8}$$

where

$$G(\lambda_1,..., \lambda_n) = \det \left| \int_{t=0}^{\infty} E_{\lambda_i}(t)\, E_{\lambda_j}(t)\, dt \right|_{i,j=1}^{n} = \det \left| \frac{1}{\lambda_i + \lambda_j} \right|_{i,j=1}^{n}.$$

Cauchy's formula [2, p. 195] then gives the explicit formula

$$G(\lambda_1,..., \lambda_n) = \prod_{i<j} (\lambda_j - \lambda_i)^2 \Big/ \prod_{i,j=1}^{n} (\lambda_j + \lambda_i)$$

which in conjunction with (8) yields (6). ∎

Note. When using generalized differential approximation with the fixed exponential basis as described in [5], the parameters $\lambda_1,..., \lambda_n$ are uniquely determined by requiring (7) to be orthogonal to all polynomials of degree $n-1$ or less with respect to the inner product associated with the measure df.

Rapid Convergence

When using Bellman's differential approximation [1, p. 226], generalized differential approximation with the fixed exponential basis [5], optimal least-squares approximation (with respect to all $2n$ parameters $a_{nj}, \lambda_{nj}, j = 1,..., n$) [4], and a number of similar algorithms, we first generate the n exponents λ_{nj}, $j = 1,..., n$, according to some rule and then determine the n linear parameters a_{nj}, $j = 1,..., n$, by a least-squares criterion. When this is done to approximate a completely monotonic function of the form (4), it can be shown that the exponents λ_{nj} are distinct points which are localized in the interval (α, β), cf. [5, Theorem 1; 4, Theorem 3 and Lemma 1]. The rapid convergence of Y_n to F is then guaranteed by the following

THEOREM. *For each $n = 1, 2,...$ let the exponents $\lambda_{n1} < \cdots < \lambda_{nn}$ lie in $[\alpha, \beta]$, where $0 < \alpha < \beta < \infty$, and let the coefficients $a_{n1},..., a_{nn}$ be chosen so as to make* (1) *the best least-squares approximation to the completely monotonic function* (4) *from the linear space spanned by $E_{\lambda_{n1}},..., E_{\lambda_{nn}}$. Then* (3) *holds with $K = (2\alpha)^{-1/2} F(0)$ and $q = (\beta - \alpha)/(\beta + \alpha)$.*

Proof. Since $a_{n1},..., a_{nn}$ are optimal in the least-squares sense, we may use (4), (5), and the lemma to write

$$\begin{aligned}
\|F - Y_n\|_2 &= \left\| \int_{\lambda=\alpha}^{\beta} E_\lambda \, df(\lambda) - Y_n \right\|_2 \\
&\leqslant \left\| \int_{\lambda=\alpha}^{\beta} [E_\lambda - \mathscr{E}_\lambda(\lambda_{n1},..., \lambda_{nn}; -)] \, df(\lambda) \right\|_2 \\
&\leqslant \int_{\lambda=\alpha}^{\beta} \|E_\lambda - \mathscr{E}_\lambda(\lambda_{n1},..., \lambda_{nn}; -)\|_2 \, df(\lambda) \\
&= \int_{\lambda=\alpha}^{\beta} (2\lambda)^{-1/2} |D_n(\lambda)| \, df(\lambda) \\
&\leqslant (2\alpha)^{-1/2} \max_{\alpha \leqslant \lambda \leqslant \beta} |D_n(\lambda)| \cdot F(0),
\end{aligned}$$

where

$$D_n(\lambda) = \prod_{j=1}^{n} [(\lambda - \lambda_{nj})/(\lambda + \lambda_{nj})]. \tag{9}$$

Finally, since $\lambda_{n1},..., \lambda_{nn}$ all lie in $[\alpha, \beta]$,

$$|D_n(\lambda)| \leqslant [(\beta - \alpha)/(\beta + \alpha)]^n$$

so the proof is complete. ∎

Note. If for each $n = 1, 2, \ldots$ we uniquely determine the roots $\lambda_{n1} < \cdots < \lambda_{nn}$ by the requirement that

$$\max_{\alpha \leqslant \lambda \leqslant \beta} |D_n(\lambda)| = \text{Minimum},$$

and then determine $a_{n1}, \ldots, a_{nn}$ by the least-squares criterion, the resulting sequence $Y_1, Y_2, \ldots$ will satisfy (3) with q replaced by some $q^* < (\beta - \alpha)/(\beta + \alpha)$. This being the case, (3) will also hold for such a q^* when Y_n is a best least-squares approximation of the form (1) to F, $n = 1, 2, \ldots$.

Note. We may also use [3, Theorem 2] to deduce a bound of the form (3) when for each $n = 1, 2, \ldots$, Y_n is a best least-squares approximation of the form (1) to a function of the form (4).

Note. When $\alpha = 0$ or $\beta = \infty$ the above argument fails. Nevertheless, if we have some means of showing that $D_n(\lambda)$ converges to 0 at each point of support of df (e.g., as is the case when the λ_{nj}'s are ultimately dense in the sense that

$$\lim_n \min_j |\lambda - \lambda_{nj}| = 0$$

at each point λ in the support of df (cf. [5, Theorem 5]) then it is still possible to establish the convergence of Y_n to F, but the rate (3) is lost in the process.

References

1. R. Bellman, "Methods of Nonlinear Analysis," Vol. I, Academic Press, New York, 1970.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. D. W. Kammler, Aproximation with sums of exponentials in $L_p[0, \infty)$, *J. Approx. Theory* **16** (1976), 384–408.
4. D. W. Kammler, Least squares approximation of completely monotonic functions by sums of exponentials, *SIAM J. Numer. Anal.* **16** (1979), 801–818.
5. D. W. Kammler, Differential approximation of completely monotonic functions, *SIAM J. Numer. Anal.* **18** (1981), 900–918.
6. R. Schaback, Suboptimal exponential approximations, *SIAM J. Numer. Anal.* **16** (1979), 1007–1018.